Raster-based Mathematical Modeling

**Map Algebra and Cartographic Modeling**
- The same algebraic notation to operate on gridded data can be used on gridded data as on single numbers. The procedure of using algebraic techniques to build models for grid-based spatial analysis is called **cartographic modeling**.
- Map algebra is numerical equivalent to conventional overlay operation. Rules of combination guides **numerical overlay operations**.
- Map algebra and cartographic modeling: The language orders the user’s thoughts and provides the rules and syntax needed for the user to communicate with the computer. In ArcInfo, you can use Arc Macro Language (AML) to develop a complicated model by combining a series of mathematical and/or Boolean operations.

**Cell-by-cell processing (point operation)**
- Worm’s eye view: only see its own location, and grid cells are seen as distinct, uncorrelated individual entities;
- All operations and functions are performed on a cell-by-cell basis.
- Operations on single cells. An operator (function) processes the first cell in an input grid and writes the result to the corresponding cell on an output grid before processing the next cell in the grid.
- Compute an output grid where output value at each location is a function of the value associated with one or more grids as that location.
- The value of the single cell, regardless of the values of the cells of its neighboring cells, has a direct influence of the value of the output.
- Functions such as the trigonometric, the exponential and logarithmic, can be applied to a single layer.
- Addition, subtraction, ratio, multiplication, linear combination, minimum, maximum, majority, or minority can be applied to multiple grids.
- Examples in ArcInfo Grid Module:
  - Grid: outgrid = ingrid1 + Ingrid2
  - Grid: outgrid = sin(ingrid1) + mean(ingrid2, ingrid3, ingrid4) + pow(ingrid5, 2)
  - Grid: outgrid = con(ingrid1 > 5, sin(ingrid1), cos(ingrid1))

**Boolean Logic Overlay Operations**
- The simplest and best-known type of GIS model.
- Using the Boolean logic rules to combine different layers.
- Procedure:
  1) Determine if a single condition (criterion) for each attribute is satisfied. Apply conditional statements to determine a binary condition based on some threshold values derived from the single criterion. The condition is evaluated as either True (1) or False (0) using reclassification of the input data layer.
  2) Combine single conditions with Boolean “AND” or “OR” operators.
  3) The resultant map is a binary 2-class map. The variable for output is either True (1) or False (0). True means that multiple criteria are simultaneously satisfied, while False (0) means that one or more conditions are not satisfied.
4) The results of this overlay procedure give a set of areas that satisfy certain criteria. The number of potentially suitable sites within the suitable area is defined; the question then becomes how to identify a limited number of best sites from the thousands available.

- Most Boolean logical based overlay procedures in GIS do not allow for the fact that variables may not be equally important and the decisions about threshold values are often subjective.

Mathematically-based Overlay Operations
- In many applications, the binary responses generated by Boolean logic operations alone are unrealistically restrictive. Mathematically based overlay allows for assigning weights to individual data layers (variables) to account for the differential impacts they might have on the final result.
- Mathematical operations allow for a more flexible combination of input data layers. Addition, subtraction, multiplication, and division, and other mathematical operations can be performed on two or more input layers to produce new output data layers.
- Map Algebra and Modeling: A language specially designed for geographic cell-based systems. The language orders the user’s thoughts and provides the rules and syntax needed for the user to communicate with the computer.
- Rules of combination guides overlay operations.

Spatial Decision Supporting System
- Spatial decision support systems (SDSS) are designed to help decision-makers solve complex spatial problems
- SDSS provide a framework for integrating
  1) Database management systems (DBMS).
  2) Graphical display capabilities.
  3) Tabular reporting capabilities.
  4) Graphic User Interface (GUI).
  5) Analysis procedures in a model base management system (MBMS).
  6) The decision-maker's expert knowledge.
- GIS is an ideal tool to construct SDSS. Most GIS normally provide 1, 2, 3, and 4, and the addition of 5 and 6 creates a SDSS.

SDSS based on Multiple Criteria Evaluation
- Real world problems for spatial analysis and decision making are inherently multi-objective in nature;
- Multiple criteria methods allow for the presence of more than one objective or goal in a complex spatial problem
- Decision makers are frequently interested in the trade-off relationship between the various criteria
- A GIS is an ideal tool to analyze and solve multiple criteria problems

MCE Based on Boolean Logic Overlay
- The simplest and best-known type of GIS-based MCE.
- Both vector and raster data structure can be used to implement Boolean logic overlays.
- In the vector implementation, buffering, intersect (AND), erase (NOT), and union (OR) operators are used in sequence.
• Most Boolean logical based overlay procedures in GIS do not allow for the fact that variables may not be equally important and the decisions about threshold values are often subjective.

**Implementation Procedure**
5) Decision maker is asked to specify goals;
6) List all influencing factors and corresponding criteria;
7) Determine if a single condition (criterion) for each attribute is satisfied. Apply conditional statements to determine a binary condition based on some threshold values derived from the single criterion (constraints). The condition is evaluated as either True (1) or False (0) using reclassification of the input data layer.
8) Combine single conditions represented by different layers with Boolean “AND”, “NOT”, or “OR” operators.
9) The resultant map is a binary 2-class map. The variable for output is either True (1) or False (0). True means that multiple criteria are simultaneously satisfied, while False (0) means that one or more conditions are not satisfied.

• This method finds the complete non-inferior solution set rather than a single solution. The results of this overlay procedure give all possible, feasible locations that satisfy certain criteria. The question then becomes how to identify the best sites from the identified suitable solution set. Final selection is left to decision-makers

**Multi-Criteria Evaluation (MCE) Based on Mathematical Overlay**
• In many applications, the binary responses generated by Boolean logic operations alone are unrealistically restrictive. Mathematically based overlay allows for assigning weights to individual data layers (variables) to account for the differential impacts they might have on the final result.
• Mathematical operations allow for a more flexible combination of input data layers. Addition, subtraction, multiplication, and division, and other mathematical operations can be performed on two or more input layers to produce new output data layers.
• Map Algebra and Modeling: A language specially designed for geographic cell-based systems. The language orders the user’s thoughts and provides the rules and syntax needed for the user to communicate with the computer.
• MCE technique allows the data layers (factors) to be weighted to reflect their relative importance and they do not rely on threshold values.
• At a conceptual level, MCE methods involve qualitative or quantitative weighting, scoring or ranking of criteria to reflect their importance to either a single or multiple set of objectives.
• Rules of combination guides overlay operations. Raster data structure is often used to implement the mathematically based MCE.

**Implementation Procedure**
1. Selection of Criteria
   • Specify the goal
   • List factors (attributes) that influence the spatial decision. The selected factors should completely describe impacts while avoiding redundancy.
   • Use some variables to quantitatively represent each factor. Sometimes, surrogate variables are required to represent influencing factors.

2. Standardization of Criterion Scores
• Standardization of the scales of measurement on each input data layer will enable meaningful comparisons to be made between the data layers.
• It is normal to apply a linear stretch routine to re-scale the values between the minimum and maximum values in a raster data layer.

3. Allocation of Weights
• Assign relative weightings for the different criteria. Weights must sum to 1.
• To assign variable weights means to make a judgment of how much of one variable to trade for another.
• The interaction and independence between variables should be considered when assigning weights to avoid the redundancy and double counting.
• Weights can be assigned and adjusted based on the judgment of experts in the domain of the application.

4. Applying the MCE algorithm.
• Input data layers can be added together using a linear weighted combination method

\[
V_j = \frac{\sum_{i=1}^{m} w_i r_{ij}}{\sum_{i=1}^{m} w_i}
\]

• The result is a value ranging between 0 and 1, which can help to determine which sites are actually the best available.

Sensitivity Analysis and Alternative Scenarios
• The above procedure is based on the assumptions:
  1) Different objectives can be expressed in commensurate terms;
  2) The problem can be collapsed and simplified into a single objective for analysis;
  3) Decision makers agree on the relative importance of the commensurable objectives;
• However, the above assumptions don't necessarily hold in the real world. For example, consensus rarely exists concerning the relationships between the various objectives
• Change weighting to assess sensitivity of solution or to reflect different opinions; these likely reflect opinions of different experts, different schools of thought, different objectives.
• A GIS generally has ideal data visualization capabilities and graphic user interface. This allows for efficient and effective visual examinations of solutions, and allows users to interactively modify weighting or threshold values to perform sensitivity analysis.

MCE based on Other Methods
1) Fuzzy Logic Method
• More flexible combination of weighted maps
• Can be readily implemented with a GIS modeling language.
• Improve on the linear additive nature of the model.
• Using fuzzy membership values (assigned subjectively) to combine
• Membership of a fuzzy set is expressed on a continuous scale from 1 (full membership) to 0 (full non-membership).
• Classes of any map can be associated with fuzzy membership values in an attribute table.
• The fuzzy membership values must reflect the relative importance of each map, as well as the relative importance of each class of a single map. The fuzzy memberships are similar to the combined effects of class scores and the map weights of the index overlay method.
• Given two or more maps with fuzzy membership functions for the same set, the fuzzy AND, fuzzy OR, fuzzy algebraic product, fuzzy algebraic sum and fuzzy gamma operator, etc. can be used to combine the membership values together.

2) Bayesian Methods
• Boolean, Index Overlay, Fuzzy logic models are subjective empirical models, with the rules, weights or fuzzy membership values being assigned subjectively, using a knowledge of the process involved to estimate the relative importance of input data layers.
• Bayesian model assign the weighting of individual data layers is based on a Bayesian probability model.
• Introduce the idea of prior and posterior (conditional) probability. The prior probability can be successively updated with addition of new evidence, so that the posterior probability from adding one piece of evidence can be treated as the prior for adding a new piece of evidence.

3) Logistical Regression Method
4) Dempster-Shafer Method